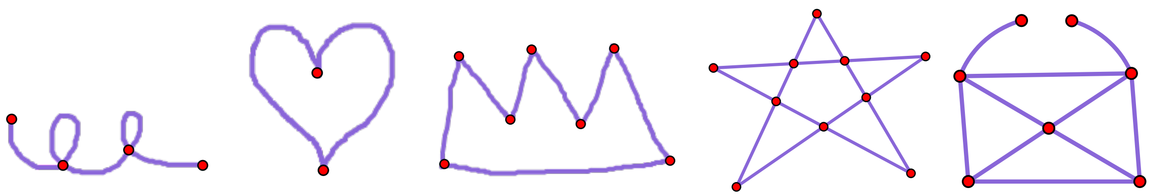
1.1 Gabriela's Doodles

1.1.1 Investigate: How Do You Doodle?

Focus Question: What stays the same and what changes in these graphs?

1) Gabriela likes to doodle during her math class. One day, she accidently left her math notebook in the classroom. Mrs. Vu, her math teacher, was curious. She opened Gabriela's notebook and found a page full of drawings like these.



What do you notice about the commonalities between these drawings?

2) Mrs. Vu noticed the following commonalities:

- In each of these doodles, there is a path from one dot to any other dot.
- Gabriela placed a dot at every intersection, corner, and endpoint.

Draw a few doodles on your own that are similar to Gabriela's. For each of your doodles, record the number of vertices, number of edges, and number of regions that the doodle splits the page into (include the space outside of your doodle as a region).

3) Suppose you want to build more doodles. Before drawing your next doodle, write the number of vertices, edges, and regions that you would like to have in your doodle.

   Number of vertices: ______________________

   Number of edges: ______________________

   Number of regions: ______________________

   Report these numbers to your teacher.

4) With a partner, decide if it is possible to build a doodle with each desired set of numbers. Keep in mind the criteria for a doodle specified in the previous problem. If it is possible, find a way to build it. If not, explain why it is not possible.

1.1.2 Describe: Programming a Computer

Focus Question: How do we systematically produce a connected graph?

Brian is trying to program his computer to generate a connected graph on the screen. The inputs for the program are the number of vertices and the number of edges. The output is a connected graph. If two edges cross, a vertex needs to be placed at the intersection. He can write commands for the computer to produce vertices and edges. Help him come up with a step-by-step procedure to accomplish this task. Show the output that the computer produces on screen for each step in your algorithm. As you work through the steps in your algorithm, think about what stays the same and what changes in your connected graphs. Test your algorithm with multiple cases.

Resource page for 1.1.2 Describe: Programming a Computer

Graph	Number of Vertices (V)	Number of Regions (R)	Number of Edges (E)

What do you notice about the relationship between the number of vertices, regions, and edges? Why do think that is true?

1.1.3 Justify: Daniela's Conjecture

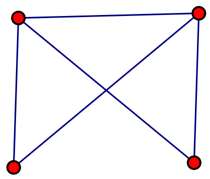
Focus Question: What is the relationship between the number of vertices, edges, and regions in a connected graph? What makes that true?

Daniela, Gabriela's twin sister, thinks that for doodles like Gabriela's, the sum of the number of vertices and the number of regions that the doodle splits the page into is two more than the number of edges in the doodle.

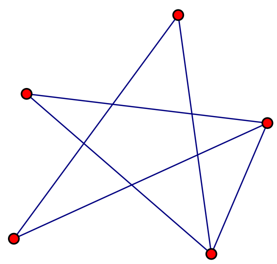
1) Verify that Daniela's conjecture is true for Gabriela's doodles.
2) Is this a coincidence? If so, show a counter-example. If not, explain what makes Daniela's conjecture true?

1.1.4 Investigate: Ana's Doodles

Ana thinks that Daniela's conjecture is wrong. "Take a look at this doodle here," she says. "It has 4 vertices, 4 regions, 5 edges, and the sum of the number of vertices and regions is 8, but two plus the number of edges is 7!"



"And it's not just one doodle," she continues, "Daniela's conjecture also doesn't hold for this doodle!"

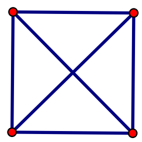


Is Daniela's conjecture still right? Or has Ana shown that it's wrong?

1.1.5 Take Notes: Euler's Formula for Planar Graphs

In the previous investigations 1.1.1 through 1.1.4, you were introduced to a special type of graph called ***planar graph***. What is a ***planar graph***?

Example: Appearances can be deceiving! The following graph is actually planar. Redraw it so that it appears planar.



Planar graphs hold a special relationship called the **Euler's Formula**:

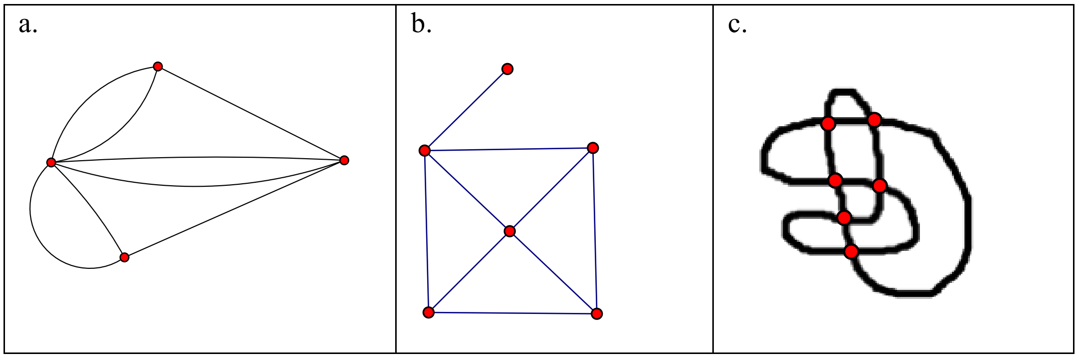
Note: The value of the expression V + R – E is called the **Euler characteristic** (or Euler number). For a planar graph, the Euler characteristic is 2. For other non-planar graphs, the Euler characteristic may not be the same.

Example: Verify that the graph above satisfies the Euler's formula.

Gabriela's Doodles - Problem Set

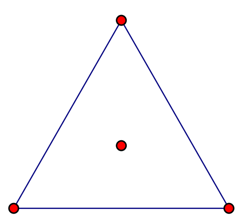
Check for Understanding

1) Describe Euler's formula for a planar graph. Give at least one example.
2) Justify why Euler's formula works.
3) Verify that Euler's formula holds for each graph shown below.



Repeated Reasoning

4) Take one of the graphs above and put a new vertex in the middle of an edge. How do the values of V, R, and E change in this case? How would the Euler characteristic, V + R – E, change?
5) Take one of the graphs above and add a new edge in the graph so that the graph remains planar. How do the values of V, R, and E change in this case? How would the Euler characteristic change? Is it possible to add an edge to an existing graph and reduce the number of regions? Why or why not?
6) Miguel was asked to verify that Euler's formula holds for the following graph. He claimed that it doesn't hold. Do you agree with Miguel? Why or why not?



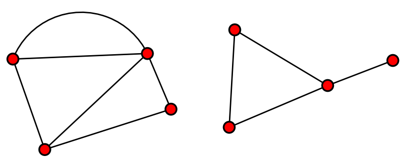
7) If a connected planar graph has 10 vertices and 15 edges, how many regions does it have?
8) If a connected planar graph has 10 regions and 21 edges, how many vertices does it have?

Diving Deeper

9) Draw two different connected graphs with 10 vertices that do not separate the plane into different regions. What is the relationship between the number of edges and vertices in each of your graphs? Generalize the relationship for any number of vertices.

10) Determine if it is possible to draw a connected graph in the plane without edge crossings with each of the following sets of conditions. If so, draw one; if not, explain why not.
    a) An even number of vertices, an even number of edges, and an odd number of regions.
    b) An odd number of vertices, an even number of edges, and an odd number of regions.
    c) An odd number of vertices, an odd number of edges, and an odd number of regions.

11) Suppose you are given a graph with two disjointed pieces shown below.



    a) What is the Euler characteristic of this graph? Explain.
    b) Suppose that you have a graph G consisting of n disconnected planar graphs. What is the Euler characteristic of G? Explain.

12) Take a rectangular box and flatten out the faces. Determine the number of regions, edges, and vertices (corners) of this graph. Does Euler's formula work for this flattened box? Why or why not?

13) Take a real-life 3D basketball. Determine the number of regions, edges, and vertices of the graph drawn on the 3D basketball (not the 2D views you see below!). Does Euler's formula work for a basketball? Why or why not?