1.1 Hidden Figures

1.1.1 Construct: Mystery Triangle

Focus question: How do I make sense of this structure? What stays the same and what changes?

Follow the steps below to uncover a mystery triangle.

Step 1: Draw a large triangle in the middle of your paper.

Step 2: Plot the midpoint of each side of the triangle.

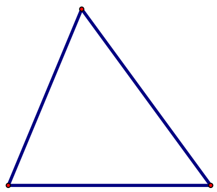
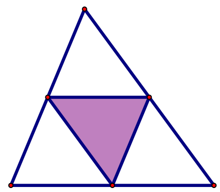
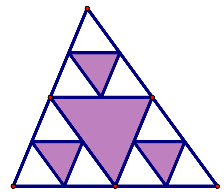
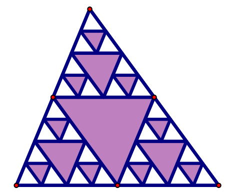
Step 3: Create 4 triangles by connecting the midpoints in step 2. Shade in the middle triangle. We claim that when you connect the three midpoints, you will construct 4 congruent triangles (why?).

Step 4: Repeat steps 2 and 3 several times on each unshaded triangle.

1.1.2 Investigate: Sierpinski Triangle

Focus questions: What patterns do you notice? What is it about the geometric structure that explains the numerical patterns you observed?

You have just created an image called the Sierpinski gasket, first introduced by Waclaw Sierpinski in 1915. The first few iterations of the Sierpinski gasket are shown below.

			
Beginning stage	1^{st} iteration	2^{nd} iteration	3^{rd} iteration

1. Is it possible to have exactly 3,003 unshaded triangles? If so, how many iterations does it take? Explain your thinking.

2. Is it possible to have exactly 9,841 shaded triangles? If so, how many iterations does it take? Explain your thinking.

3. Determine the fraction of the area of the original triangle that the shaded triangles take up after 7 iterations. What about after any number of iterations? Explain your thinking.

4. If the pattern continues forever, what will the total area of the shaded triangles eventually be? Explain your thinking.

5. Suppose that the perimeter of the original triangle is 1. Determine the total perimeters of all shaded triangles after 7 iterations. What about after any number of iterations? Explain your thinking.

1.1.3 Construct: Mystery Curve

Focus question: How do I make sense of this structure? What stays the same and what changes?

For this task, please use a pencil, an eraser, and a ruler. [Note: **Do not use a pen.**]

Follow the steps below to uncover the mystery curve.

Step 1: Draw a horizontal segment about 6 inches long in the middle of your paper. Leave about 3 inches of space above the line.

Step 2: Divide the line segment into three congruent pieces.

Step 3: Draw an equilateral triangle that has the middle piece as its base; then erase this piece.

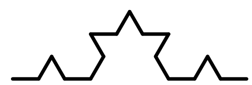
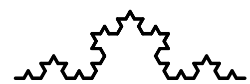
Step 4: Repeat steps 2 and 3 on each of the four pieces of the curve.

Step 5: Repeat steps 2 & 3 at least one more time on each new segment resulted from the previous step.

1.1.4 Investigate: Koch Curve

Focus questions: What patterns do you notice? What is it about the geometric structure that explains the numerical patterns you observed?

You just created a curve called the Koch curve, first introduced by Helge von Koch in 1904. The first few stages of the Koch curve are shown below.

			
Beginning Stage	1st iteration	2nd iteration	3rd iteration

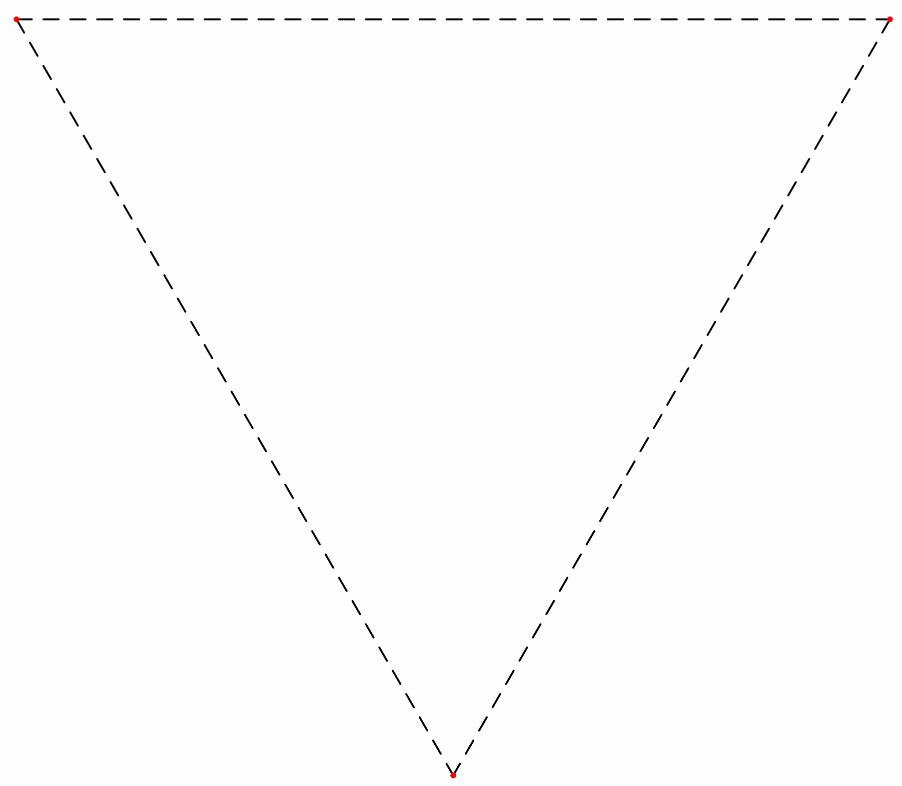
1) Is it possible to have exactly 1,000,000 segments at some point? If so, how many iterations does it take? If not, explain why not.

2) Suppose that the length of the segment at the beginning stage is 6 inches. What is the length of the curve after 4 iterations? Explain.

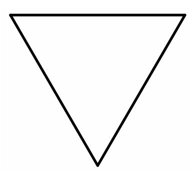
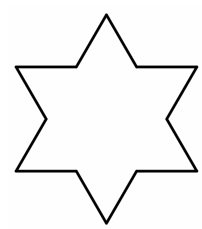
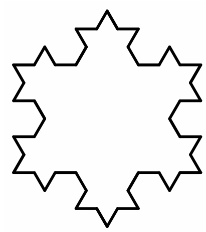
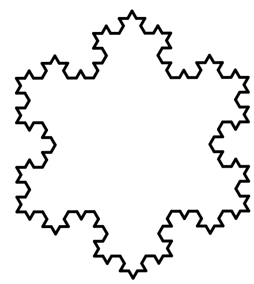
3) If the pattern in this curve is repeated forever, what would be eventually the length of the curve?

1.1.5 Extend: Koch Snowflake

1) For each side of the equilateral triangle shown below, apply at least two iterations of the same pattern in the previous Koch Curve.



The same curve can be repeated on each side of an equilateral triangle to create this beautiful snowflake.

Beginning Stage	1st iteration	2nd iteration	3rd iteration
			

2) Suppose that the length of each side the original equilateral triangle is 1 unit.

a. Determine the perimeter, P_n, of the Koch snowflake for any number of iterations, n.

b. Determine the area, A_n, of the Koch snowflake for any number of iterations, n.

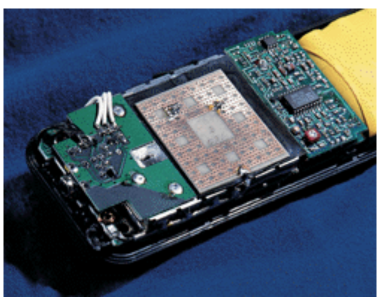
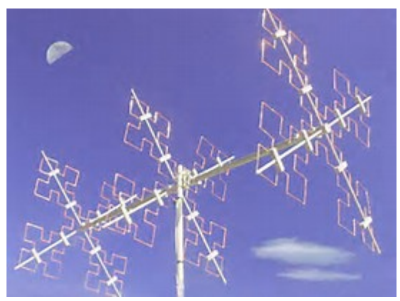
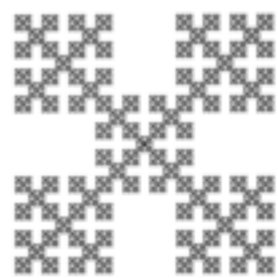
c. If the pattern in this curve is repeated forever, what eventually will be the perimeter and the area of the snowflake?

1.1.6 Apply: Fractal Antenna

Focus question: How does math apply in everyday life?

A fractal, based on the Latin word ***fractus*** meaning "broken", is a fragmented geometric shape that can be subdivided in parts. Fractal is considered self-similar; that is, the figure looks much the same no matter how close or how far it is viewed, and parts of it is a reduced-size copy of the whole. We can trace the origins of fractal theory to Helge von Koch in the early 1900s. The term ***fractal*** was coined by French mathematician Benoît Mandelbrot in 1975. Well-known mathematical structures that are fractals include Sierpinski's gasket, Cantor's comb, von Koch's snowflake, the Mandelbrot set, et al. Fractals are used to model many real-world objects and phenomena such as clouds, rock formations of mountains, coastlines, human circulatory system and turbulence in fluids.

Self-similarity in fractal design is used in many mobile devices and antenna to maximize the length or increase the perimeter (on inside sections or the outer structure), of material that can receive or transmit electromagnetic radiation within a given total surface area or volume. Fractal antennas, first created by Nathan Cohen in 1995, are compact, multiband or wideband, and have useful applications in cellular telephone and microwave communications.

 http://www.scienceprog.com/wp-content/uploads/2007i/fractal/cell_antenna.gif	 http://www.vk6fh.com/vk6fh/fractal1.PNG	 Integrated circuit fractal antenna in a hearing aid device https://www.google.com/patents/US6710744

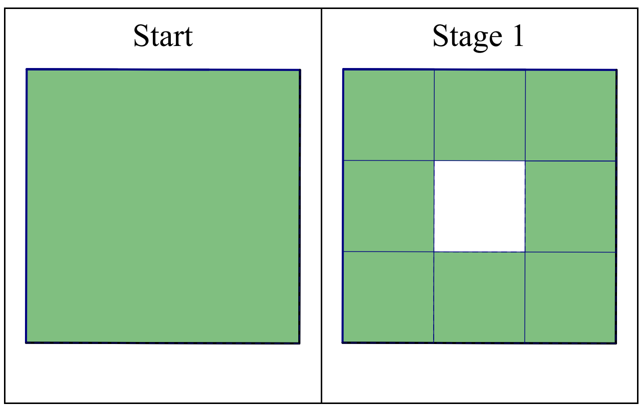
Extension: Find other real-life examples in nature, architecture, music, technology etc. where fractal geometry is used.

Hidden Figures – Problem Set

Checking for Understanding

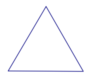
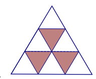
1. Sierpinski carpet: The following set of steps describes a variation of the Sierpinski gasket.
   Start with a solid square.
   Step 1: Divide the square into 9 equal smaller squares and remove the center small square.
   Step 2: On every remaining solid square, repeat step 1. Repeat this process over and over for subsequent stages.

   Here are sample drawings of the starting step and the figure at stage 1.



   a. Using graph paper, draw the figures for two more stages.
   b. Determine the number of square holes for stages 1 through 4, stage 10, and any stage n.
   c. Suppose the number of square holes for stage n is H_n, express the number of square holes for stage $n + 1$ in terms of H_n.
   d. Suppose the length of each side of the starting square is 3 units. Determine the shaded areas of the figure for stages 1 through 4, stage 10, and stage n.
   e. Suppose the total shaded area of the figure for stage n is An, express the shaded area for stage n + 1 in terms of An.
   f. Suppose the length of each side of the starting square is 3 units. Determine the length of the boundary of the figure for stages 1 through 4, stage 10, and stage n.
   g. Suppose the length of the boundary of the figure is Bn, express the area for stage n + 1 in terms of Bn.
   h. If the process is continued forever, what will eventually be the amount of shaded area and the length of the boundary? Explain.

Repeated Reasoning

2. Here is another variation of the Sierpinski gasket.
   - Start with an equilateral triangle.
   - For each empty triangle , replace it with  .

   Carefully construct the starting triangle and at least 3 more stages. Answer the following questions.

   a. How many solid (shaded) triangles are in each stage from stage 1 through stage 4? Stage n?

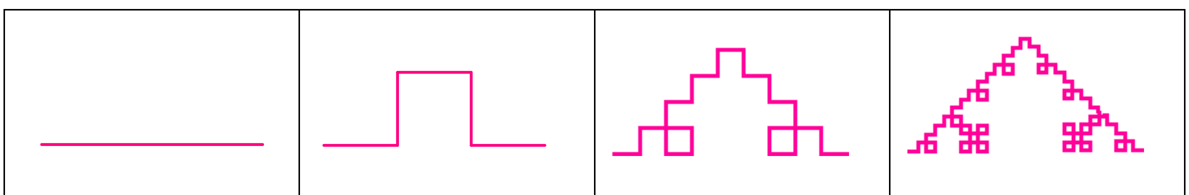
   b. If the perimeter of the starting triangle is 1, what is the total perimeters of all solid triangles in each stage, from stage 1 through stage 4? Stage n?

   c. If the total perimeters of all shaded triangles in stage n is P_n, what is the total perimeters of all shaded triangles in stage $n + 1$?

   d. If the area of the starting triangle is 1, what is the total area of all solid triangles in each stage, from stage 1 through stage 4? Stage n?

   e. If the process is continued forever, what will eventually be the total area of the solid (shaded) part? What about the total perimeters of all shaded triangles? Explain.

3. The Hat Curve is formed by dividing a line segment into three congruent pieces. The middle piece is then replaced by three sides of a square (with sides of length equal to the removed middle piece of the original segment). The process is then repeated on each of the five pieces of the curve.



Stage 1	Stage 2	Stage 3	Stage 4

   a. Find the number of segments in each of the first five stages.

   b. Is it possible to have exactly 10,000 segments in some stage? If so, what is the stage number? If not, explain why not.

   c. How many segments are in stage 11? Stage 89?

   d. If you know the number of segments in a certain stage, how you would find the number of segments in the next stage? In the stage that is 6 steps after it?

e. If you know the number of segments in a certain stage, how you would find the number of segments in the stage before it? In the stage that is 7 steps before it?

f. If we denote f(n) the number of sides in the nth stage, explain what f(5) represents?

g. Cyrus thinks that he has found a formula to express the number of segments in a stage:

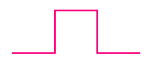
h. Miley found another formula to express the number of segments in a stage:

i. Whose formula is correct? Explain.

j. Given the stage number n, how would you determine the number of segments in that stage? Explain.

k. What is the total number of segments in the first 100 stages in the sequence?

Diving Deeper

4. Quadratic Koch snowflake: The Quadratic Koch snowflake is constructed as follows.
   - Start from a square.
   - Whenever you see a segment, replace with the Hat curve .

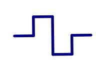
Carefully construct the starting square (stage 0) and at least 2 more stages. Assume that the side length of the starting square is 1. Answer the following questions.

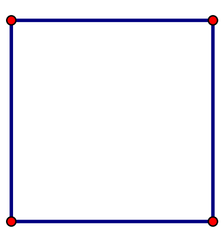
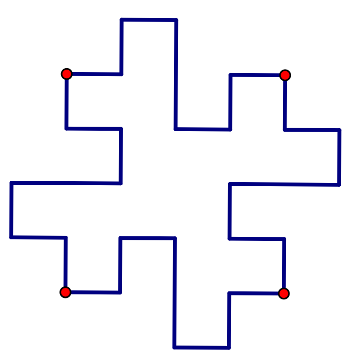
a. What is the perimeter of the figure in stage 1? Stage 2? Stage 5? Stage n?

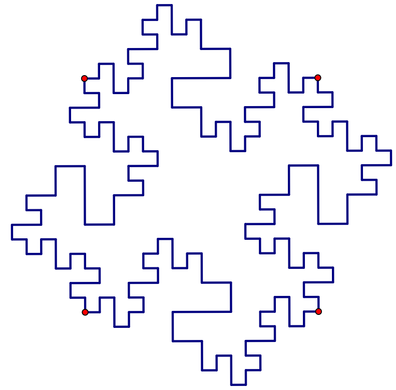
b. What is the area of the figure in stage 1? Stage 2? Stage 5? Stage n?

c. If the process is continued forever, what will eventually be the area of the figure? What about the total perimeters of all shaded triangles. Explain.

5. The Koch Island is constructed as follows:

Start with a square. Whenever you see a segment , replace it with .

Stage 0	Stage 1	Stage 2
		

Suppose that original square has sides of length 1.

a. What is the perimeter of the figure in stage 1? Stage 2? Stage 5? Stage n?

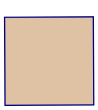
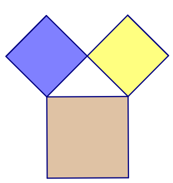
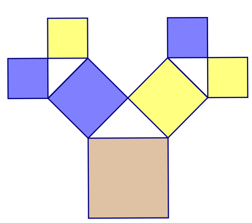
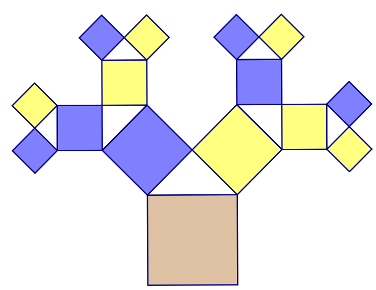
b. What is the area of the figure in stage 1? Stage 2? Stage 5? Stage n?

c. If the pattern is continued forever, what happens to its perimeter and area?

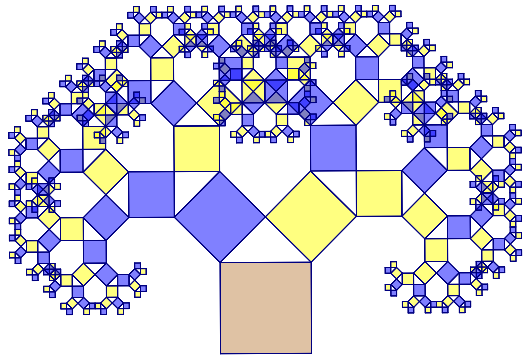
6. The following fractal is called the Pythagorean tree which was first drawn by Albert E. Bosman (1891-1961) around 1942. Bosman was a Dutch electrical engineering and mathematics teacher. In 1957, he published the book Het wondere onderzoekingsveld der vlakke meetkunde ("The Wondrous Exploration Field of Plane Geometry") that contained his illustrations of the Pythagorean tree.

Here is one way to construct the Pythagorean tree.

- Begin with a square.
- Construct a right isosceles triangle (or any right triangle) whose hypotenuse is the top edge of the square.
- Construct squares along each of the other two sides of this isosceles triangle.
- Repeat this construction recursively on each of the two new squares. The figures below show the three iterations.

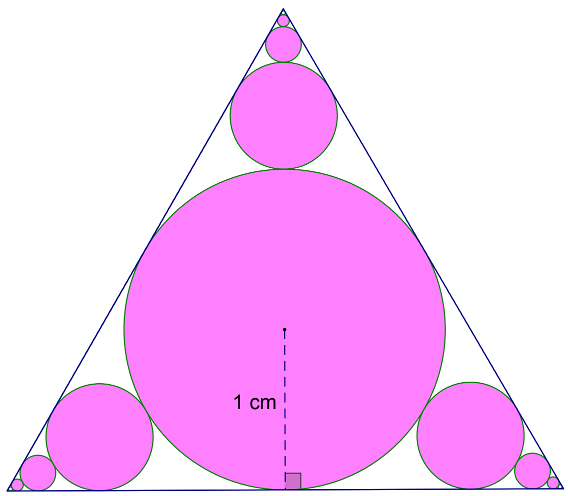
Stage 0	Stage 1	Stage 2	Stage 3
			

After many, many iterations, we get this tree.



a. Determine the number of squares for any stage, n.
b. If the length of the original square is 1 unit and we follow one branch, either the right-most or the left-most, determine the length of the top-most square at any stage, n.

7. A circle of radius 1 cm is inscribed in an equilateral triangle. A smaller circle is inscribed at each vertex touching the first circle and tangent to the two of the three sides of the triangle. Repeat this process indefinitely.



a. What is the sum of the circumferences of all the circles?

b. What is the sum of their areas?

c. Adding all the circumferences or adding all the areas, which sum grows faster?

8. Create your own fractal image. It can be a variation of the fractal curve you have seen in this the previous lessons or something totally original. Write at least two mathematical questions about your fractal. Provide a detailed solution for each question.